

GLOBAL OKOUNKOV BODIES FOR BOTT-SAMELSON VARIETIES

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ABSTRACT. We use the theory of Mori dream spaces to prove that the global Okounkov body of a Bott-Samelson variety with respect to a natural flag of subvarieties is rational polyhedral. In fact, we prove more generally that this holds for any Mori dream space which admits a flag of Mori dream spaces satisfying a certain regularity condition. As a corollary, Okounkov bodies of effective line bundles over Schubert varieties are shown to be rational polyhedral. In particular, it follows that the global Okounkov body of a flag variety G/B is rational polyhedral.

As an application we show that the asymptotic behaviour of dimensions of weight spaces in section spaces of line bundles is given by the counting of lattice points in polytopes.

INTRODUCTION

Okounkov bodies were first introduced by A. Okounkov in his famous paper [Oko96] as a tool for studying multiplicities of group representations. The idea is that one should be able to approximate these multiplicities by counting the number of integral points in a certain convex body in \mathbb{R}^n . More precisely, the setting is the following. Let G be a complex reductive group which acts as automorphisms on an effective line bundle L over a projective variety X , and hence defines a representation on the space of sections $H^0(X, L^k)$ for each integral power, L^k , of L . Okounkov constructs a convex compact set $\Delta \subseteq \mathbb{R}^n$, where $n = \dim X$, with the property that for each irreducible finite-dimensional representation V_λ , where λ —the so-called *highest weight*—is a parameter, the multiplicity $m_{k\lambda, k} := \dim \operatorname{Hom}_G(V_{k\lambda}, H^0(X, L^k))$ of $V_{k\lambda}$ in $H^0(X, L^k)$ is asymptotically given by the volume of the convex body $\Delta_\lambda := \Delta \cap H_\lambda$, where $H_\lambda \subseteq \mathbb{R}^{n+1}$ is a certain affine subspace, in the following sense:

$$(1) \quad \lim_{k \rightarrow \infty} \frac{m_{k\lambda, k}}{k^m} = \operatorname{vol}_m(\Delta_\lambda),$$

where m is the dimension of Δ_λ , and the volume on the right hand side denotes the m -dimensional Euclidean volume of Δ_λ . An approximation of the integral $\operatorname{vol}_m(\Delta_\lambda)$ by Riemann sums yields that the multiplicity $m_{k\lambda, k}$ is asymptotically given by the number of points of the set $\Delta_\lambda \cap \frac{1}{k}\mathbb{Z}^m$.

The construction of the body Δ is purely geometric and depends on a choice of a flag Y_\bullet , $Y_n \subseteq Y_{n-1} \subseteq \cdots \subseteq Y_0 = X$ of irreducible subvarieties of X , and the “successive orders of vanishing” of certain invariant sections

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$s \in H^0(X, L^k)$ along this flag. It was later realized by Kaveh and Khovanskii ([KK09]), and independently by Lazarsfeld and Mustață, ([LM09]), that Okounkov's construction makes sense for more general subseries of the section ring $R(X, L)$ of a line bundle over a variety X , and that the asymptotics of dimensions of linear series can be studied by counting lattice points of convex compact bodies. For example, the analog of (1) for the complete linear series of a big line bundle L is given by the identity

$$\lim_{k \rightarrow \infty} \frac{h^0(X, L^k)}{n!k^n} = \frac{1}{n!} \text{vol}_n(\Delta_{Y_\bullet}(L)),$$

where $\Delta_{Y_\bullet}(L)$ denotes the Okounkov body of the line bundle L with respect to the flag Y_\bullet .

The above formula shows in particular that the volume of the Okounkov body is an invariant of the line bundle L , and thus does not depend on the choice of the flag Y_\bullet . However, the shape of $\Delta_{Y_\bullet}(L)$ depends heavily on the flag, and it is a notoriously hard problem to explicitly describe these bodies, or even to show that they possess some nice properties, such as being polyhedral. A yet more difficult problem is to determine the global Okounkov body $\Delta_{Y_\bullet}(X)$ of a variety X (cf. [LM09]), which is a convex cone in a certain Euclidean space, and no longer depends on a particular line bundle L .

Returning to the original motivation by Okounkov of studying multiplicities of representations, there is also another approach to describing multiplicities by counting lattice points in convex bodies, namely Littelmann's construction of string polytopes ([Li98]). The setting here is the following. Let G , again, be a complex reductive group, and let $H \subseteq G$ be a maximal torus in G . Then any irreducible finite-dimensional G -representation V_λ admits a basis of weight vectors with respect to H , and this basis is parameterized by the integral points in a rational polytope C^λ , the *string polytope* of V_λ . Moreover, the approximative lattice counting problem is even exact here. Since the irreducible representations V_λ can be realized as section spaces $H^0(X, L_\lambda)$, where $X = G/B$ for a Borel subgroup $B \subseteq G$, and L_λ is a line bundle over X , it would be interesting to recover Littelmann's string polytopes C^λ as Okounkov bodies, or at least to construct rational polyhedral Okounkov bodies which describe asymptotic multiplicities of weight spaces.

In the present paper we study both problems described above—namely the Okounkov bodies for complete linear series, and the asymptotics of weight multiplicities—for general Bott-Samelson varieties $Z = Z_w$ (given by a reduced expression w for an element \bar{w} in the Weyl group of G), that is, Bott-Samelson varieties which desingularize some Schubert variety X_w in a flag variety G/B .

Our main result is formulated in the more general context of Mori dream spaces. It is well known that a Mori dream space X admits finitely many small \mathbb{Q} -factorial modifications such that any movable divisor D on X is the pullback of a nef divisor under one of these modifications. In particular, for any effective divisor E , all necessary flips in the E -MMP exist and terminate. We define an admissible flag $X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$ to be *good*, if

- (1) Y_i is a Mori dream space for each $0 \leq i \leq n$, and for $i = 1, \dots, n$, Y_i is cut out by a global section $s_i \in H^0(Y_{i-1}, \mathcal{O}_{Y_{i-1}}(Y_i))$, and
- (2) for any small \mathbb{Q} -factorial modification $f : Y_i \dashrightarrow Y'_i$, the exceptional locus $\text{exc}(f)$ intersects Y_{i+1} properly.

Our main result is the following.

Theorem A. *Let X be a Mori dream space and assume that there exists a good flag $Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_n$. Then the global Okounkov body $\Delta_{Y_\bullet}(X)$ of X with respect to the flag Y_\bullet is a rational polyhedral cone.*

In order to apply the above result to Bott-Samelson varieties, we prove the following.

Theorem B. *Let $Z = Z_w$ be a Bott-Samelson variety defined by a reduced expression w of an element \bar{w} in the Weyl group of G .*

- (i) *The variety Z is log-Fano and hence a Mori dream space. (Theorem 2.1)*
- (ii) *There exists naturally a good flag Y_\bullet on Z , the so called vertical flag. (Proposition 4.1)*

As a consequence, all Okounkov bodies $\Delta_{Y_\bullet}(L)$ of line bundles L over Z are rational polyhedral. Using the desingularization $Z_w \rightarrow X_w$ of the Schubert variety X_w , we also see that line bundles over Schubert varieties admit rational polyhedral Okounkov bodies.

On the representation-theoretic side, we obtain Okounkov bodies describing weight multiplicities. Indeed, the flag Y_\bullet of subvarieties is B -invariant, which allows for the construction of affine subspaces H_μ mentioned before. We then get the following result on asymptotics of weight multiplicities in a section ring $R(Z, L)$.

Theorem C. *Let L be an effective line bundle over the Bott-Samelson variety Z . Let $H \subseteq B$ be a torus contained in a maximal torus of G lying in B , and let μ be an rational H -weight. Then there exists an affine subspace H_μ (in \mathbb{R}^{n+1} , where $n = \dim Z$) such that the asymptotics of the multiplicity function $m_{k\mu, k}$ defined above is given by*

$$\lim_{k \rightarrow \infty} \frac{m_{k\mu, k}}{k^m} = \text{vol}_m(\Delta_{Y_\bullet}(L) \cap H_\mu),$$

where $\Delta_{Y_\bullet}(L)$ is the rational polyhedral Okounkov body of L .

If we apply this to the situation when the torus H is a maximal torus, Z is of maximal dimension, and thus admits a birational morphism $f : Z \rightarrow G/B$ to the flag variety of G , and $L = f^*(L_\lambda)$ is the pull-back of the line bundle L_λ over G/B , we obtain the following corollary, which can be seen as an analog of Littelmann's result which describes weight multiplicities using string polytopes.

Corollary D. *Let $V_\lambda \cong H^0(G/B, L_\lambda)$ be the irreducible G -representation of highest weight λ . If μ is rational weight, let $m_{k\mu, k}$, for $k\mu$ integral, denote the multiplicity of the weight $k\mu$ in the G -module $V_{k\lambda}$. Then there exists an $m \in \mathbb{N}$ such that*

$$\lim_{k \rightarrow \infty} \frac{m_{k\mu, k}}{k^m} = \text{vol}_m(\Delta_{Y_\bullet}(f^*(L_\lambda)) \cap H_\mu).$$

The present paper is organized as follows: we begin by recalling basic facts about Okounkov bodies and Bott-Samelson varieties in sections 1 and 2, respectively. The main result is proved in section 3. Finally, in section 4 the result is applied to Bott-Samelson varieties, furthermore we address there the representation-theoretic consequences.

We work throughout over the complex numbers \mathbb{C} as our base field.

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1. OKOUNKOV BODIES

For the convenience of the reader not familiar with the construction of Okounkov bodies we give here a quick overview. For a thorough discussion, we refer the reader to [KK09] and [LM09].

To a graded linear series W_\bullet on a normal projective variety X of dimension n we want to assign a convex subset of \mathbb{R}^n carrying information on W_\bullet . In practice, more often than not, W_\bullet will be the complete graded linear series $\bigoplus_k H^0(X, \mathcal{O}_X(kD))$ corresponding to an effective divisor D . However, at times it is convenient to work with an arbitrary linear series associated to some divisor D , i.e., a series $W_\bullet = \{W_k\}$ of subspaces $W_k \subseteq H^0(X, \mathcal{O}_X(kD))$ satisfying the condition $W_k \cdot W_l \subseteq W_{k+l}$. The construction will depend on the choice of a valuation-like function

$$\nu : \bigsqcup_{k \geq 0} W_k \setminus \{0\} \longrightarrow \mathbb{Z}^n.$$

Instead of recounting the conditions on ν , we describe a certain type of valuation which automatically satisfies these conditions. To this end, let

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n$$

be a flag of irreducible subvarieties such that $\text{codim}_X(Y_i) = i$ and such that Y_n is a smooth point of each Y_i . Then, for a section $s \in W_k \subseteq H^0(X, \mathcal{O}_X(kD))$, we set $\nu_1(s) := \text{ord}_{Y_1}(s)$. If we choose a local equation for Y_1 , we obtain a unique section $\tilde{s}_1 \in H^0(X, \mathcal{O}_X(kD - \nu_1(s)Y_1))$ not vanishing identically along Y_1 , and thus determining a section $s_1 \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1(s)Y_1))$. We then set $\nu_2(s) = \text{ord}_{Y_2}(s_1)$, and proceed as before to obtain the valuation vector $\nu_{Y_\bullet}(s) = (\nu_1(s), \dots, \nu_n(s))$.

One then defines the valuation semi-group of W_\bullet with respect to Y_\bullet as

$$\Gamma_{Y_\bullet}(W_\bullet) := \{(\nu_{Y_\bullet}(s), m) \in \mathbb{Z}^{n+1} \mid 0 \neq s \in W_m\}.$$

Furthermore, we define the Okounkov body of W_\bullet as

$$\Delta_{Y_\bullet}(W_\bullet) := \Sigma(\Gamma_{Y_\bullet}(W_\bullet)) \cap (\mathbb{R}^n \times \{1\})$$

where $\Sigma(\Gamma_{Y_\bullet}(W_\bullet))$ denotes the closed convex cone in \mathbb{R}^{n+1} spanned by $\Gamma_{Y_\bullet}(W_\bullet)$.

If W_\bullet is the complete graded linear series of a divisor D , we write $\Delta_{Y_\bullet}(D)$ for the Okounkov body of W_\bullet . In this case, by [LM09, Theorem 2.3], we have the important identity

$$\mathrm{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(D)) = \frac{1}{n!} \mathrm{vol}_X(D),$$

showing in particular, that the volume of the body $\Delta_{Y_\bullet}(D)$ is independent of the choice of the flag Y_\bullet . Another important observation made in [LM09] is that even the shape of the Okounkov body $\Delta_{Y_\bullet}(D)$ for a divisor D only depends on the numerical equivalence class of D . It is therefore a natural question how the bodies $\Delta_{Y_\bullet}(D)$ change as $[D]$ varies in the Néron-Severi vector space $N^1(X)_{\mathbb{R}}$. An answer to this question is given in [LM09, Theorem 4.5] by proving the existence of the global Okounkov body: there exists a closed convex cone

$$\Delta_{Y_\bullet}(X) \subset \mathbb{R}^n \times N^1(X)_{\mathbb{R}}$$

such that for each big divisor D the fiber of the second projection over $[D]$ is exactly $\Delta_{Y_\bullet}(D)$.

The concrete determination or even the description of geometric properties of Okounkov bodies associated to some graded linear series is extremely difficult in general. As is to be expected this will be even more true of the global Okounkov body of a given variety. In particular, it is an intriguing question under which conditions on X it is possible to pick a flag such that the corresponding global Okounkov body is rational polyhedral. Already in [LM09] this was shown to be possible for toric varieties. Based on this evidence it is conjectured to work also for any Mori dream space X . In [SS14], we introduce a possible technique to prove rational polyhedrality of global Okounkov bodies by constructing so-called Minkowski bases on X . Work in progress hints at the feasibility of this approach for any Mori dream space. In this paper however, we use a more direct strategy to prove the rational polyhedrality of global Okounkov bodies with respect to a natural choice of flag for a special class of Mori dream spaces, namely Bott-Samelson varieties, which we introduce in the following section.

Let us make a small remark on how the construction by Littelmann mentioned in the introduction compares to Okounkov bodies. Littelmann's string polytopes are constructed by purely algebraic and combinatorial means, notably using quantum enveloping algebras of Lie algebras, and the result thus only shows formal analogies with the outcome of Okounkov's approach. However, since—by the Borel-Weil theorem—every irreducible G -module V_λ can be realized as the space of sections $H^0(X, L_\lambda)$ of a line bundle L_λ over a flag variety $X := G/B$, where B is a Borel subgroup of G , Okounkov's approach makes sense for the study of asymptotics of weight spaces in the section ring $R(X, L_\lambda)$. For the approach to work, the flag Y_\bullet should consist of H -invariant subvarieties. A natural candidate for such a flag would then be a flag of Schubert varieties, and indeed this approach was taken by Kaveh in [K11]. For technical reasons, notably for having a flag of Cartier divisors, Kaveh passed to a Bott-Samelson resolution $Z \rightarrow X$ of X , pulled back L_λ to Z , and replaced the flag Y_\bullet by a flag Z_\bullet of (translations of) Bott-Samelson subvarieties of Z . The main result in [K11] is that Littelmann's string bases can be interpreted in terms of a \mathbb{Z}^n -valued valuation

on the function field $\mathbb{C}(Z)$ of Z , depending on the flag Z_\bullet . This valuation, however, differs from those introduced by Okounkov: whereas orders of vanishing of a regular function f are described in local coordinates x_1, \dots, x_n by the smallest monomial term of $f(x) = \sum_{a \in \mathbb{N}^n} c_a x^a$, with respect to some ordering of the variables x_1, \dots, x_n , Kaveh's valuation is locally defined by the highest monomial term. In geometric language, this valuation thus tells how often f can be differentiated in the various directions defined by the x_i in the given order. It still remains an open problem to interpret Littelmann's string polytopes as Okounkov bodies, or indeed, more generally, to construct rational polyhedral Okounkov bodies for line bundles over flag varieties using some H -invariant flag Y_\bullet .

2. BOTT-SAMELSON VARIETIES

Let us recall the basics of Bott-Samelson varieties, following [LT04].

Let G be a connected complex reductive group, let $B \subseteq G$ be a Borel subgroup, and let W be the Weyl group of G . If $s_i \in W$ is a simple reflection, let P_i denote the associated minimal parabolic subgroup containing B . Then the quotient space P_i/B is isomorphic to \mathbb{P}^1 . For a sequence $w = (s_1, \dots, s_n)$ (where the s_i are not necessarily distinct), let $P_w := P_1 \times \dots \times P_n$ be the product of the corresponding parabolic subgroups, and consider the right action of B^n on P_w given by

$$(p_1, \dots, p_n)(b_1, \dots, b_n) := (p_1 b_1, b_1^{-1} p_2 b_2, b_2^{-1} p_3 b_3, \dots, b_{n-1}^{-1} p_n b_n).$$

The Bott-Samelson variety Z_w is the quotient

$$Z_w := P_w / B^n.$$

An alternative description of this quotient can be given as follows. Suppose that X and Y are two varieties, such that X is equipped with a right action and Y with a left action of B . Consider the right action of B on the product given by

$$(x, y).b := (xb, b^{-1}y), \quad (x, y) \in X \times Y, \quad b \in B,$$

and let $X \times_B Y := (X \times Y)/B$ denote the quotient space. Then the map $X \times_B Y \rightarrow X/B$, $[(x, y)] \mapsto xB$ exhibits $X \times_B Y$ as a fiber bundle over X/B and with fiber Y . Now, we can alternatively describe Z_w as

$$Z_w = (P_1 \times_B \dots \times_B P_n) / B,$$

where B acts on the right on $P_1 \times_B \dots \times_B P_n$ by

$$[(p_1, \dots, p_n)].b := [(p_1, \dots, p_{n-1}, p_n b)], \quad (p_1, \dots, p_n) \in P_w, \quad b \in B.$$

As a consequence, using the fact that each quotient P_i/B is isomorphic to \mathbb{P}^1 , Z_w is given as an iteration of \mathbb{P}^1 -bundles. To describe this structure in more detail, let, for $j \in \{1, \dots, n\}$, $w[j]$ denote the truncated sequence (s_1, \dots, s_{n-j}) , and let $Z_{w[j]} := P_{w[j]} / B^{n-j}$ denote the associated Bott-Samelson variety. Then the projections $P_w \rightarrow P_{w[j]}$ are B^n -equivariant, where B^n acts on $P_{w[j]}$ by the factor B^{n-j} , and thus induce a projections $\pi_{w[j]} : Z_w \rightarrow Z_{w[j]}$, which can be factorized as a sequence of \mathbb{P}^1 -fibrations

$$\pi_{w[1]} : Z_w \rightarrow Z_{w[1]} \rightarrow \dots \rightarrow Z_{w[j]}.$$

Let $\pi : Z_w \longrightarrow P_1/B$ denote the composition of all these projections, i.e., π is the projection morphism onto P_1/B defined by the description of Z_w as the bundle

$$Z_w := P_1 \times_B (P_2 \times_B \cdots \times_B P_n).$$

Now, each \mathbb{P}^1 -bundle admits a natural section as follows. Let $w(j) := (s_1, \dots, \hat{s}_j, \dots, s_n)$, so that $P_{w(j)}$ embeds naturally as a subgroup of P_w . The embedding $\sigma_{w,j}^0 : P_{w(j)} \longrightarrow P_w$ is B^{n-1} -equivariant, and thus induces an embedding

$$\sigma_{w,j} : Z_{w(j)} \longrightarrow Z_w$$

of $Z_{w(j)}$ as a divisor in Z_w such that the divisors $Z_{w(j)}$, $j = 1, \dots, n$, intersect transversely in a point. In particular, $\sigma_{w,n} : Z_{w(n)} \cong Z_{w[1]} \longrightarrow Z_w$ defines a section of the \mathbb{P}^1 -bundle $\pi_{w[1]} Z_w \longrightarrow Z_{w[1]}$, and identifies $Z_{w[1]}$ with a divisor which is transversal to the fibers of $\pi_{w[1]}$. Now, the Picard group $\text{Pic}(Z_w)$ splits as the direct sum

$$\text{Pic}(Z_w) \cong \text{Pic}(Z_{w[1]}) \oplus \mathbb{Z},$$

where \mathbb{Z} is identified with the subgroup generated by the line bundle $\mathcal{O}_{Z_w}(Z_{w(n)})$. Iterating the above splitting yields that the line bundles $\mathcal{O}_{Z_w}(Z_{w(j)})$, $j = 1, \dots, n$, define a basis for $\text{Pic}(Z_w)$. Clearly, they are all effective. Conversely, if w is a reduced sequence, i.e., if the length of the product $\overline{w} := s_1 \cdots s_n$ equals n , a divisor

$$m_1 Z_{w(1)} + \cdots + m_n Z_{w(n)}, \quad m_1, \dots, m_n \in \mathbb{Z},$$

is effective if and only if $m_1, \dots, m_n \geq 0$ (cf. [LT04, Prop. 3.5]). The basis $\{Z_{w(1)}, \dots, Z_{w(n)}\}$ is called the *effective basis* for $\text{Pic}(Z_w)$. Notice that, since $Z_{w(n)}$ defines a section of the bundle $\pi_{w[1]}$, the restricted divisors

$$(2) \quad Z_{w(1)} \cdot Z_{w(n)}, \dots, Z_{w(n-1)} \cdot Z_{w(n)}$$

form the effective basis for $Z_{w[1]} \cong Z_{w(n)}$.

2.1. The vertical flag. We also recall the so-called $\mathcal{O}(1)$ -basis for $\text{Pic}(Z_w)$ defined as follows. Each product $P_1 \times \cdots \times P_k$, $k = 1, \dots, n$, defines a morphism

$$\varphi_k : Z_{w[n-k]} \longrightarrow G/B, \quad [(p_1, \dots, p_k)] \mapsto p_1 p_2 \cdots p_k B.$$

Put $\mathcal{O}_{w[n-k]}(1) := \varphi_k^* L_{\omega_\alpha}$, where α is simple root corresponding to the simple reflection s_k , ω_α is its fundamental weight, viewed as a character of B , and $L_{\omega_\alpha} := G \times_B \mathbb{C}$ is the associated line bundle over the flag variety G/B . Let $\mathcal{O}_k(1) := \pi_{w[n-k]}^* \mathcal{O}_{w[n-k]}(1)$. The line bundles $\mathcal{O}_1(1), \dots, \mathcal{O}_n(1)$, being pullbacks of globally generated line bundles, are then globally generated and form a basis for $\text{Pic}(Z_w)$. Moreover, a line bundle $\mathcal{O}_1(1)^{m_1} \otimes \cdots \otimes \mathcal{O}_n(1)^{m_n}$ is very ample (nef) if and only if $m_1, \dots, m_n > 0$ ($m_1, \dots, m_n \geq 0$) ([LT04, Thm. 3.1]). Notice here that the morphism φ_k above is induced by the B -equivariant multiplication map

$$P_1 \times \cdots \times P_k \longrightarrow G, \quad (p_1, \dots, p_k) \mapsto p_1 p_2 \cdots p_k,$$

where B acts on $P_1 \times \cdots \times P_k$ by the right multiplication on the factor P_k . We can therefore also view the line bundle $\mathcal{O}_{w[n-k]}(1)$ over $Z_{w[n-k]}$ as the product

$$\mathcal{O}_{w[n-k]}(1) = P_1 \times \cdots \times P_k \times_{B^k} \mathbb{C},$$

where B^k acts on the product $P_1 \times \cdots \times P_k$ by

$$(p_1, \dots, p_k) \cdot (b_1, \dots, b_k) := (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{k-1}^{-1} p_k b_k),$$

and on \mathbb{C} by the character

$$(b_1, \dots, b_k) \mapsto \omega_\alpha(b_k).$$

Thus, the sections of the sheaf $\mathcal{O}_{w[n-k]}(1)$ over an open set $U \subseteq Z_{w[n-k]}$ correspond to the regular functions f on the B^k -invariant open subset

$$\tilde{U}_k := \{(p_1, \dots, p_k) \in P_1 \times \cdots \times P_k \mid [(p_1, \dots, p_k)] \in U\}$$

satisfying the B^k -equivariance property

$$\begin{aligned} f(p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{k-1}^{-1} p_k b_k) &= \omega_\alpha(b_k)^{-1} f(p_1, \dots, p_k), \\ (p_1, \dots, p_k) &\in \tilde{U}_k, \quad (b_1, \dots, b_k) \in B^k. \end{aligned}$$

It follows that sections of the sheaf $\mathcal{O}_k(1)$ over the open subset $\pi_{w[n-k]}^{-1}(U)$ then correspond to the regular functions f on the B^n -invariant open subset

$$\tilde{U} := \{(p_1, \dots, p_n) \in P_1 \times \cdots \times P_n \mid [(p_1, \dots, p_k)] \in U\}$$

satisfying the B^n -equivariance property

$$\begin{aligned} f(p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n) &= \omega_\alpha(b_n)^{-1} f(p_1, \dots, p_n), \\ (p_1, \dots, p_n) &\in \tilde{U}, \quad (b_1, \dots, b_n) \in B^n. \end{aligned}$$

In other words, $\mathcal{O}_k(1)$ is the line bundle

$$\mathcal{O}_k(1) = P_1 \times \cdots \times P_n \times_{B^n} \mathbb{C},$$

where B^n acts on \mathbb{C} by the character

$$\xi_k : B^n \rightarrow \mathbb{C}^\times, \quad \xi_k(b_1, \dots, b_n) := \omega_\alpha(b_k).$$

Since the $\mathcal{O}_k(1)$, $k = 1, \dots, n$, form a basis for the Picard group, we see that each line bundle corresponds to a unique character $\xi_1^{m_1} \cdots \xi_n^{m_n}$, for an $(m_1, \dots, m_n) \in \mathbb{Z}^n$ in such a way that the sections of the sheaf $\mathcal{O}_1(1)^{m_1} \otimes \cdots \otimes \mathcal{O}_n(1)^{m_n}$ over an open subset $U \subseteq Z_w$ correspond to the regular functions f on the B^n -invariant open subset

$$\tilde{U} := \{(p_1, \dots, p_n) \in P_1 \times \cdots \times P_n \mid [(p_1, \dots, p_k)] \in U\}$$

satisfying the B^n -equivariance property

$$\begin{aligned} f(p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n) &= \xi_1(b_1)^{-m_1} \cdots \xi_n(b_n)^{-m_n} f(p_1, \dots, p_n), \\ (3) \quad (p_1, \dots, p_n) &\in \tilde{U}, \quad (b_1, \dots, b_n) \in B^n. \end{aligned}$$

Thus, every line bundle L on Z_w is of the form

$$L = P_1 \times \cdots \times P_n \times_{B^n} \mathbb{C},$$

where B^n acts on \mathbb{C} by the character $\xi_1^{m_1} \cdots \xi_n^{m_n}$, for a unique $(m_1, \dots, m_n) \in \mathbb{Z}^n$. Let $\mathcal{O}(m_1, \dots, m_n)$ denote the line bundle corresponding to (m_1, \dots, m_n) .

In particular, each line bundle $L = \mathcal{O}(m_1, \dots, m_n)$ admits an action of P_1 as bundle automorphisms by

$$p \cdot [(p_1, \dots, p_n), z] \mapsto [(pp_1, p_2, \dots, p_n), z],$$

which is clearly well-defined since the left multiplication of P_1 on the P_1 -factor in $P_1 \times \dots \times P_n$ commutes with the B^n -action on this product. The induced representation of P_1 on the space of global sections $H^0(Z_w, \mathcal{O}(m_1, \dots, m_n))$ is given by

$$(4) \quad (p \cdot f)(p_1, \dots, p_n) := p \cdot (f(p^{-1}p_1, p_2, \dots, p_n)),$$

for $p \in P_1$ and $(p_1, \dots, p_n) \in P_1 \times \dots \times P_n$, where we have used the identification (3) of sections with equivariant regular functions on $P_1 \times \dots \times P_n$. Clearly, the B^n -equivariance property is preserved by the left action of P_1 , so that this indeed defines a representation of P_1 on $H^0(Z_w, \mathcal{O}(m_1, \dots, m_n))$.

Consider now the fibration

$$\pi : Z_w \longrightarrow P_1/B \cong \mathbb{P}^1$$

given by mapping $[(p_1, \dots, p_n)]$ to the class $[p_1] = p_1 \cdot B$. All of its fibers are isomorphic to the Bott-Samelson variety $Z_{[s_2, \dots, s_n]}$. Note that B operates on the quotient P_1/B as the upper triangular matrices acts on \mathbb{P}^1 ,

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} [z_0 : z_1] = [az_0 + bz_1 : cz_1],$$

and thus with exactly one fixed point p_0 . Denote by Y the fiber over p_0 . Note also that for any $p \in P_1$ and $[p_1, \dots, p_n] \in Z_w$ we have

$$\begin{aligned} \pi(p \cdot [p_1, \dots, p_n]) &= \pi([pp_1, p_2, \dots, p_n]) \\ &= pp_1 B = p \cdot \pi([p_1, \dots, p_n]), \end{aligned}$$

i.e., π is P_1 -equivariant, and hence P_1 acts as automorphisms of the fiber bundle π .

We can reiterate this construction to obtain a natural flag of Bott-Samelson varieties

$$Z_w \supseteq Y_1 \supseteq \dots \supseteq Y_n$$

where Y_i is given as the fiber of the corresponding \mathbb{P}^1 -bundle $Y_{i-1} \longrightarrow \mathbb{P}^1$ over the B -fixed point. We call this flag the *vertical flag* on Z_w .

2.2. Bott-Samelson varieties as Mori dream spaces. Now assume that w is a reduced sequence. Recall that the product map $P_w \longrightarrow G, (p_1, \dots, p_n) \longrightarrow p_1 \dots p_n$ induces a morphism

$$p_w : Z_w \longrightarrow Y_{\overline{w}} := \overline{B\overline{w}B}$$

into the Schubert subvariety $Y_{\overline{w}}$ of the flag variety G/B , and that this morphism is in fact birational. Moreover, it is B -equivariant with respect to the left action of B on Z_w defined by

$$b[(p_1, \dots, p_n)] := [(bp_1, p_2, \dots, p_n)], \quad (p_1, \dots, p_n) \in P_w, \quad b \in B.$$

In particular, if \overline{w} is the longest element of the Weyl group, p_w defines a birational map $Z_w \longrightarrow G/B$.

The following theorem is the crucial result for our application of the theory from the following section.

Theorem 2.1. *Let G be a complex reductive group with Weyl group W , and let $Z = Z_w$ be a Bott-Samelson variety defined by a reduced sequence w of simple reflections. Then Z admits a divisor Δ such that (Z, Δ) is a log Fano pair. In particular, Z is a Mori dream space.*

Proof. Let $Y = G/B$, and let D_ρ be the divisor on $Y_{\overline{w}}$ which corresponds to the restriction to $Y_{\overline{w}}$ of the square root of the anticanonical bundle of Y . Then D_ρ is an ample divisor on $Y_{\overline{w}}$, so that $p_w^*(D_\rho)$ is a nef divisor on Z .

In order to facilitate the notation, let $\{D_1, \dots, D_n\}$ be the basis of effective divisors for $\text{Pic}(Z)$. Now choose integers $a_1, \dots, a_n > 0$ so that $\sum_{i=1}^n a_i D_i$ is an ample divisor. Then, for every $N > 0$, the divisor $p_w^*(-D_\rho) - \sum_{i=1}^n a_i/N D_i$ is anti-ample. Now let $N \in \mathbb{N}$ be so big that $a_i/N < 1$ for every i , and put

$$\Delta := \sum_{i=1}^n (1 - a_i/N) D_i.$$

If K_Z is the canonical divisor of Z , we then have that

$$K_Z + \Delta = \pi^*(-D_\rho) - \sum_{i=1}^n D_i + \sum_{i=1}^n (1 - a_i/N) D_i = \pi^*L_{-\rho} - \sum_{i=1}^n (a_i/N) D_i$$

(cf. [LT04, Lemma 5.1]) is anti-ample. Since Z is nonsingular, and all subsets of the set of smooth divisors $\{D_1, \dots, D_n\}$ intersect transversely and smoothly, the pair (Z, Δ) thus defines a log Fano pair. \square

Remark 2.2. In the context of the above theorem it is worth mentioning an analogous result by Anderson and Stapledon ([AS14]) on the log Fano property of Schubert varieties.

3. GOOD FLAGS ON MORI DREAM SPACES

In this section we prove the main theorem of this paper. The main objective is to establish conditions on a flag on a Mori dream space, such that its global Okounkov body is rational polyhedral.

First let us recall that a Mori dream space X is a normal \mathbb{Q} -factorial variety such that $\text{Pic}(X)_{\mathbb{Q}} \cong N^1(X)_{\mathbb{Q}}$ and with a Cox ring $\text{Cox}(X)$ which is a finitely generated \mathbb{C} -algebra. We make use of the theory of Mori dream spaces developed by Hu and Keel in [HK00] and we refer the reader to this beautiful paper for a detailed investigation of Mori dream spaces.

Note that for any effective divisor D on a Mori dream space X , the ring of sections $R(X, D) := \bigoplus_{k \geq 0} H^0(X, \mathcal{O}_X(kD))$ is finitely generated, so we obtain a natural rational map

$$f_D : X \dashrightarrow \text{Proj}(R(X, D))$$

which is regular outside the stable base locus of D . One obtains an equivalence relation of effective divisors as follows: two effective divisors D and D' are *Mori-equivalent* if up to isomorphism they yield the same rational maps. Hu and Keel prove ([HK00, Proposition 1.11]) that there are only finitely many equivalence classes, indexed by contracting rational maps $f : X \dashrightarrow X'$ and that the closure Σ_f of a maximal dimensional equivalence class can be described as the closed convex cone spanned by f -exceptional rays together with the face $f^*(\text{Nef}(X'))$ of the moving cone. These subcones Σ_f , which

decompose the pseudo-effective cone $\overline{\text{Eff}}(X)$, are in the remainder of this paper following [HK00] referred to as *Mori-chambers*.

Now, let $f : X \dashrightarrow X'$ be a small \mathbb{Q} -factorial modification defining an isomorphism

$$f|_U : U \rightarrow V$$

between open subsets $U \subseteq X, V \subseteq X'$ with complements of codimension at least two. Then f^* induces an isomorphism of pseudo-effective cones $\overline{\text{Eff}}(X) \cong \overline{\text{Eff}}(X')$ as well as an isomorphism $\text{Cox}(X) \cong \text{Cox}(X')$ of Cox rings. We further recall that f is induced by GIT in the following manner.

Let $R = \text{Cox}(X)$. The variety X can be written as the GIT-quotient $X = \text{Spec}(R)^{ss}(\chi)/G$, where G is the complex torus of rank equal to the Picard number of X , and $\text{Spec}(R)^{ss}(\chi)$ denotes the set of semistable points in $\text{Spec}(R)$ with respect to the character χ of G . We even have that $\text{Spec}(R)^{ss}(\chi) = \text{Spec}(R)^s(\chi)$; the set of χ -semistable points equals the set of χ -stable points, so that the quotient is even a geometric quotient. Also X' is a geometric quotient of the set of stable points with respect to a character χ' : $X' = \text{Spec}(R)^s(\chi')/G$. Let $\pi_\chi : \text{Spec}(R)^s(\chi) \rightarrow X$ and $\pi_{\chi'} : \text{Spec}(R)^s(\chi') \rightarrow X'$ denote the respective quotient morphisms. In both cases the sets of unstable points (with respect to χ and χ') are of codimension at least two in $\text{Spec}(R)$, and the rational map f is induced on the level of quotients by the inclusion

$$\text{Spec}(R)^s(\chi) \cap \text{Spec}(R)^s(\chi') \subseteq \text{Spec}(R)^s(\chi')$$

of the subset of common stable points into the set of χ' -stable points. In particular, the exceptional locus of f equals the complement of the domain of definition of f and is given as the image of the χ' -unstable and χ -stable points;

$$\text{exc}(f) = \pi_\chi(\text{Spec}(R)^s(\chi) \cap \text{Spec}(R)^{us}(\chi')),$$

so that f induces an isomorphism

$$\begin{aligned} f|_U : U &\xrightarrow{\cong} V, \quad U := \pi_\chi(\text{Spec}(R)^s(\chi) \cap \text{Spec}(R)^s(\chi')), \\ V &:= \pi_{\chi'}(\text{Spec}(R)^s(\chi) \cap \text{Spec}(R)^s(\chi')). \end{aligned}$$

Now let $f : X \dashrightarrow X'$ be a small \mathbb{Q} -factorial modification as above, and assume that $Y \subseteq X$ is an irreducible hypersurface given as the zero set $Y = Z(s)$ of a section $s \in H^0(X, L)$ of some line bundle L . Let $L' := (f^{-1})^*L$ denote the corresponding line bundle on X' , let $s' \in H^0(X', L')$ be the section of L' corresponding to s , and put $Y' := Z(s')$. The restriction of f to Y then defines a birational map

$$f_Y : Y \dashrightarrow Y',$$

yielding the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \uparrow & & \uparrow \\ Y & \xrightarrow{f_Y} & Y' \end{array}$$

where the vertical arrows denote the respective inclusion morphisms. We now have the following simple but useful lemma.

Lemma 3.1. *a) The hypersurface $Y' \subseteq X'$ is irreducible.*

b) If Y intersects the exceptional locus $\text{exc}(f)$ properly, then f_Y induces an isomorphism of open subsets

$$U \cap Y \xrightarrow{\cong} V \cap Y',$$

where $\text{codim}_Y(Y \setminus (U \cap Y)) \geq 2$.

Proof. Consider the restriction $(\pi_\chi)_Z : Z \rightarrow Y$ of π_χ to the preimage $Z := \pi_\chi^{-1}(Y) \subseteq \text{Spec}(R)^s(\chi)$ of Y . The fibers of $(\pi_\chi)_Z$, being closed G -orbits, are irreducible, and they are all of the same dimension, $\dim G$, since all stabilizers of points in $\text{Spec}(R)^s(\chi)$ are finite. Since Y is irreducible, it then follows that Z is irreducible, and hence also the closure \overline{Z} of Z in $\text{Spec}(R)$. Moreover, \overline{Z} can be written as the zero-set $\overline{Z} = V(f)$, where we identify the section $s \in H^0(X, L) \subseteq R$ with a function f on $\text{Spec}(R)$. Thus, the zero-set $V(f) \subseteq \text{Spec}(R)$ is irreducible. Now, the zero set $Y' = Z(s')$ is given as $Z(s') = \pi_{\chi'}(V(f) \cap \text{Spec}(R)^s(\chi'))$ and is clearly also irreducible. This proves a).

As for b), the argument above shows that f_Y in general defines an isomorphism

$$U \cap Y = U \cap Z(s) \xrightarrow{\cong} V \cap Z(s') = V \cap Y'.$$

The condition that Y intersect the exceptional locus $\text{exc}(f)$ properly then shows that the codimension of $Y \setminus (U \cap Y)$ in Y is at least two. This finishes the proof. \square

We now turn to Okounkov bodies on a Mori dream space X equipped with an admissible flag Y_\bullet . Our strategy is to deduce properties of the global Okounkov body of X from those of Okounkov bodies of line bundles restricted to Y_1 and to argue inductively. We are thus particularly interested in a comparison of the Okounkov bodies of a graded linear series coming from restricting sections to Y_1 and of a restricted line bundle. More concretely, for a divisor D on X we consider the restriction map

$$R : \bigoplus_k H^0(X, \mathcal{O}_X(kD)) \rightarrow \bigoplus_k H^0(Y, \mathcal{O}_Y(kD \cdot Y))$$

and hope for an identity

$$(5) \quad \Delta_{Y_\bullet^1}(\text{im } R) = \Delta_{Y_\bullet^1}(D \cdot Y),$$

where Y_\bullet^1 denotes the flag $Y_2 \supseteq \dots \supseteq Y_n$ on Y_1 . Note that in case D is ample the above identity holds. This follows from the exact sequence

$$H^0(X, \mathcal{O}_X(mD)) \rightarrow H^0(Y, \mathcal{O}_Y(mD)) \rightarrow H^1(X, \mathcal{O}_X(mD - Y))$$

together with the fact that for large m the last cohomology group is trivial by Serre's vanishing theorem. In order to get the desired identity for any movable divisor D , we will consider the corresponding small modification.

Proposition 3.2. *Let X be a Mori dream space, and let Y_\bullet be an admissible flag of normal subvarieties, write $Y := Y_1$, and let Y_\bullet^1 denote the admissible flag*

$$Y_n \subseteq \dots \subseteq Y_1 = Y$$

of subvarieties of Y .

Let $f : X \dashrightarrow X'$ be a SQM, and let $f_Y : Y \dashrightarrow Y'$ be the induced birational morphism. Assume that there exist open subsets $U \subset Y$ and $V \subset Y'$ with $\text{codim}_Y(Y \setminus U) \geq 2$ such that $f_Y : U \rightarrow V$ is an isomorphism.

If D' is a nef divisor on X' and $D := f^*(D')$ is the corresponding divisor on X , then the identity

$$\Delta_{Y^\bullet}(im R) = \Delta_{Y^\bullet}(D \cdot Y)$$

of Okounkov bodies holds.

Proof. Since $S := im R$ is a graded linear subseries of $\bigoplus_k H^0(Y, \mathcal{O}_Y(kD \cdot Y))$, the claimed identity of Okounkov bodies follows if we can show that both series have the same volume.

We have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \uparrow & & \uparrow \\ Y & \xrightarrow{f_Y} & Y' \end{array}$$

where the vertical arrows denote the respective inclusion morphisms. By assumption, there are open subsets $U \subset Y$ and $V \subset Y'$ with $\text{codim}_Y(Y \setminus U) \geq 2$ such that $f_Y : U \rightarrow V$ is an isomorphism.

Let us first assume that D' is ample. Denote the restricted divisors $D \cdot Y$ and $D' \cdot Y'$ by D_Y and $D'_{Y'}$, respectively.

Since D defines the map f , we have the identity

$$D = f^*(\mathcal{O}_{X'}(1)).$$

This restricts to U as the identity $f_Y^*(\mathcal{O}_{Y'}(1))|_U = D_Y|_U$. By assumption, $Y \setminus U$ has codimension at least 2 in Y , so that we get

$$f_Y^*(\mathcal{O}_{Y'}(1)) = D_Y.$$

Being ample, $\mathcal{O}_{Y'}(1)$ does not have a divisorial base component in $Y' \setminus V$, so we can represent it as a divisor $A = \overline{A} \cap \overline{V}$. Therefore, under f_Y^{-1} the divisor D_Y pulls back to A .

Since Y' might not be normal we consider the normalization

$$\pi : \widetilde{Y'} \rightarrow Y'.$$

Note that π defines an isomorphism $\pi^{-1}(V) \rightarrow V$ since V is contained in the normal locus of Y' . In particular, we have the identity

$$(f_Y^{-1} \circ \pi)^*(D_Y) = \pi^* f_Y^{-1*}(D_Y) = \pi^*(A).$$

Since $f_Y^{-1} \circ \pi$ is a contracting birational map between normal varieties, this implies the identity of volumes

$$\text{vol}(D_Y) = \text{vol}(\pi^*(A)).$$

Since π is a birational morphism, the right hand side is just $\text{vol}(A)$. On the other hand,

$$\text{vol}(A) = \text{vol}(\mathcal{O}_{Y'}(1)) = \text{vol}(S),$$

since $Y' = \text{Proj}(S)$. This proves the claim in case D' is ample.

If D' is merely nef, write $[D']$ as a limit $[D'] = \lim_{i \rightarrow \infty} [D'_i]$ of numerical equivalence classes of ample divisors $D'_i, i \in \mathbb{N}$. Put $D_i := f^* D'_i$. Then $[D] = \lim_{i \rightarrow \infty} [D_i]$. Now, let $a \in \Delta_{Y_\bullet}(D \cdot Y)$. Then $(a, [D \cdot Y]) \in \Delta_{Y_\bullet}(Y)$. Now choose points $a_i \in \Delta_{Y_\bullet}(Y \cdot D_i)$ so that $(a, [D \cdot Y]) = \lim_{i \rightarrow \infty} (a_i, [D_i \cdot Y])$. By the above, the identity (5) holds when D is replaced by $D_i, i \in \mathbb{N}$. Hence, by [LM09, Theorem 4.26],

$$((0, a_i), [D_i]) \in \Delta_{Y_\bullet}(X)$$

for each i , so that

$$((0, a), [D]) = \lim_{i \rightarrow \infty} ((0, a_i), [D_i]) \in \Delta_{Y_\bullet}(X),$$

i.e., $(0, a) \in \Delta_{Y_\bullet}(D)$. This shows that the identity (5) holds for an arbitrary nef divisor D' on X' . \square

In order to apply the above proposition to obtain information on the structure of the global Okounkov body of a Mori dream space, we need the following construction formulated in a more general context. Here Y_\bullet can be any admissible flag on a normal projective variety X .

If $s \in H^0(X, \mathcal{O}_X(m_1 D_1 + \dots + m_n D_n))$ is a section which does not vanish on Y_1 , so that $\nu(s) = (\nu_1(s), \dots, \nu_n(s))$ with $\nu_1(s) = 0$, then the restriction of s to Y_1 defines a section of the line bundle $\mathcal{O}_{Y_1}(D \cdot Y_1)$ over Y_1 with value $\nu^1(s) = (\nu_2(s), \dots, \nu_n(s))$ with respect to the truncated flag

$$(6) \quad Y_n \subseteq \dots \subseteq Y_1$$

on Y_1 .

For a finite set F_1, \dots, F_r of movable divisors on X , let

$$\Gamma(F_1, \dots, F_r) \subseteq \text{Mov}(X)$$

be the semigroup generated by the divisors F_1, \dots, F_r , and let

$$C(F_1, \dots, F_r) \subseteq \text{Mov}(X)$$

be the cone generated by F_1, \dots, F_r . Define the semigroups

$$S(F_1, \dots, F_r) := \{(\nu(s), [D]) \in \mathbb{N}_0^n \times \Gamma(F_1, \dots, F_r) \mid s \in H^0(X, \mathcal{O}_X(D)), \\ D \in \Gamma(F_1, \dots, F_r), \nu_1(s) = 0\}$$

and

$$S_1(F_1, \dots, F_r) := \{(\nu^1(s), [D \cdot Y_1]) \in \mathbb{N}_0^{n-1} \times N^1(Y_1)_{\mathbb{R}} \mid [D] \in \Gamma(F_1, \dots, F_r), \\ s \in H^0(Y_1, \mathcal{O}_{Y_1}(D \cdot Y_1))\},$$

as well as the morphism of semigroups

$$q_0 : S \rightarrow S_1, \quad q_0(\nu(s), [D]) := (\nu^1(s), [D \cdot Y_1]),$$

which extends to the linear map

$$(7) \quad q : \quad \mathbb{R}^n \oplus V(F_1, \dots, F_r) \longrightarrow \mathbb{R}^{n-1} \oplus N^1(Y_1)_{\mathbb{R}}, \\ ((x_1, \dots, x_n), [D]) \mapsto ((x_2, \dots, x_n), [D \cdot Y_1]),$$

where $V(F_1, \dots, F_r) \subseteq N^1(X)_{\mathbb{R}}$ is the \mathbb{R} -vector space generated by the numerical equivalence classes $[F_1], \dots, [F_r]$. Furthermore, we denote by $C(S(F_1, \dots, F_r))$ and $C(S_1(F_1, \dots, F_r))$ the closed convex cones in $\mathbb{R}^n \times$

$N^1(X)_{\mathbb{R}}$ and $\mathbb{R}^{n-1} \times N^1(Y_1)_{\mathbb{R}}$ spanned by the semigroups $S(F_1, \dots, F_r)$ and $S_1(F_1, \dots, F_r)$, respectively.

We now recall that for a Mori dream space X the pseudo-effective cone $\overline{\text{Eff}}(X)$ is the union of finitely many Mori chambers, $\Sigma_1, \dots, \Sigma_m$, where each Mori chamber Σ_j is the convex hull of finitely many integral divisors $D_1^j, \dots, D_{\ell_j}^j$. More concretely, by [HK00, Proposition 1.1], the chambers are in correspondence to contracting birational maps $f_j : X \dashrightarrow X_j$ with image a Mori dream space, and are given as the convex cone spanned by $f_j^*(\text{Nef}(X_j))$ together with the rays spanning the exceptional locus $\text{exc}(f_j)$. The corresponding decomposition of a divisor $D \in \Sigma_j$ is exactly its decomposition into its fixed and movable parts. We can thus reorder the divisors spanning each chamber in such a way that the first n_j of them are movable and the remaining ones are fixed. Let $\sigma_i^j \in H^0(X, \mathcal{O}_X(N_i^j))$ be the defining section of D_i^j , for $j = 1, \dots, m$, $i = n_j + 1, \dots, \ell_j$.

Corollary 3.3. *Let Y_{\bullet} be an admissible flag on a Mori dream space X such that the conclusion of Proposition 3.2 holds for any SQM $f : X \dashrightarrow X'$ and any nef divisor $D' \subset X'$. Let furthermore D_1, \dots, D_r those generators of a Mori chamber Σ which are movable. Then we have the identity*

$$(8) \quad C(S(D_1, \dots, D_r)) \\ = q^{-1}(C(S_1(D_1, \dots, D_r))) \cap (\{0\} \times \mathbb{R}_{\geq 0}^{n-1} \times C(D_1, \dots, D_r)).$$

Proof. This follows from the condition together with the fact that there exists a SQM $\pi : X \dashrightarrow X'$ such that each divisor in the cone $C(D_1, \dots, D_r)$ is a pullback by π of a nef divisor on X' ([HK00, Proposition 1.11(3)]). \square

We can now prove the following theorem which will—together with identity (8)—enable us to inductively infer information on the shape of global Okounkov bodies of certain Mori dream spaces.

Theorem 3.4. *Suppose in the above situation that each of the cones $C(S(D_1^j, \dots, D_{n_j}^j))$ is rational polyhedral with generators given by vectors $w_1^j, \dots, w_{r_j}^j$. Then the global Okounkov body $\Delta_{Y_{\bullet}}(X)$ is the cone generated by the vectors*

$$(9) \quad (\nu(s_{Y_1}, [Y_1]), (\nu(\sigma_i^j), [D_i^j]), w_h^j,$$

for

$$j = 1, \dots, m, \quad i = n_j + 1, \dots, \ell_j, \quad h = 1, \dots, r_j.$$

Proof. Let E be an effective integral divisor on X , and let $s \in H^0(X, \mathcal{O}_X(E))$ be a nonzero section of E . Let $\nu_1(s) = a$. Then, $\zeta := s/s_{Y_1}^a$, where $s_{Y_1} \in H^0(X, \mathcal{O}_X(Y_1))$ is the defining section of Y_1 , is a section of $\mathcal{O}_X(E - aY_1)$ which vanishes to order 0 along Y_1 . Now, let $\Sigma = \text{conv}\{D_1, \dots, D_{\ell}\}$ be a Mori chamber such that $E - aY_1 \in \Sigma$, and with generators ordered so that D_1, \dots, D_r are the movable generators. Let $E - aY_1 = P + N$ be the corresponding decomposition of $E - aY_1$ into its movable part P and fixed part N . Choose $M \in \mathbb{N}$ large enough such that all the divisors $MP = c_1D_1 + \dots + c_{\ell}D_{\ell}$, $MN = c_{r+1}D_{r+1} + \dots + c_{\ell}D_{\ell}$, where $c_1, \dots, c_{\ell} \in \mathbb{N}_0$, and all c_iD_i are integral divisors. Let $\sigma_i \in H^0(X, \mathcal{O}_X(D_i))$ be the defining

section of N_i , $i = r + 1, \dots, \ell$. The section $\zeta^M \in H^0(X, \mathcal{O}_X(m(E - aY_1)))$ now decomposes uniquely as a product

$$\zeta^M = \eta\sigma,$$

where $\eta \in H^0(X, \mathcal{O}_X(c_1D_1 + \dots + c_rD_r))$, and $\sigma = \sigma_{r+1}^{c_{r+1}} \dots \sigma_\ell^{c_\ell}$. Since $\nu_1(\zeta) = 0$, we also have $\nu_1(\eta) = 0$. Now, by assumption we have integral generators $w_1, \dots, w_k \in \mathbb{R}^n \times \overline{\text{Eff}}(X)$ for the cone $C(S(D_1, \dots, D_r))$, so that $(\nu(\eta), MP) = s_1w_1 + \dots + s_kw_k$, for some $s_1, \dots, s_k \geq 0$. Hence,

$$\begin{aligned} (\nu(s), E) &= a(\nu(s_{Y_1}), Y_1) + \frac{c_{r+1}}{M}(v(\sigma_{r+1}), [D_{r+1}]) + \dots + \frac{c_\ell}{M}(v(\sigma_\ell), [D_\ell]) \\ &\quad + \frac{s_1}{M}w_1 + \dots + \frac{s_k}{M}w_k. \end{aligned}$$

It follows that $\Delta_{Y_\bullet}(X)$ lies in the closed convex cone generated by the vectors (9). Since all these vectors clearly belong to $\Delta_{Y_\bullet}(X)$, this finishes the proof. \square

We are now in the position to prove the main result of this paper. Let us first define what we mean by a good flag on a Mori dream space.

Definition 3.5. Let X be a Mori dream space. An admissible flag $X = Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_n$ is *good*, if

- (1) Y_i is a Mori dream space for each $0 \leq i \leq n$, and for $i = 1, \dots, n$, Y_i is cut out by a global section $s_i \in H^0(Y_{i-1}, \mathcal{O}_{Y_{i-1}}(Y_i))$, and
- (2) for any small \mathbb{Q} -factorial modification $f : Y_i \dashrightarrow Y'_i$, the exceptional locus $\text{exc}(f)$ intersects Y_{i+1} properly.

Our main theorem now follows from the above results.

Theorem 3.6. Assume that the Mori dream space X admits a good flag Y_\bullet . Then $\Delta_{Y_\bullet}(X)$ is rational polyhedral.

Proof. We prove the theorem by induction over n . Every Mori dream curve is isomorphic to \mathbb{P}^1 , which for any choice of flag (i.e., choice of a point) has rational polyhedral global Okounkov body, namely the cone in \mathbb{R}^2 spanned by the points $(0, 1)$ and $(1, 1)$.

For the inductive step assume that $\Delta_{Y_\bullet^1}(Y_1)$ is rational polyhedral. By Theorem 3.4, what we need to prove is that for any Mori chamber Σ in $\overline{\text{Eff}}(X)$ the set of movable generators D_1, \dots, D_r of Σ yield a rational polyhedral cone $C(S(D_1, \dots, D_r))$.

Since Y_\bullet is a good flag, in particular the assumptions of Proposition 3.2 are satisfied for $Y_1 \subset X$, so we can apply Corollary 3.3. Since the linear map q (cf. (7)) is defined over \mathbb{Z} , equality (8) implies that $C(S(D_1, \dots, D_r))$ is rational polyhedral if $C(S_1(D_1, \dots, D_r))$ is. Now the rational polyhedrality of $C(S_1(D_1, \dots, D_r))$ follows from the rational polyhedrality of $\Delta_{Y_\bullet^1}(Y_1)$ since

$$C(S_1(D_1, \dots, D_r)) = pr_2^{-1}(\Gamma(D_1 \cdot Y_1, \dots, D_r \cdot Y_1) \cap \Delta_{Y_\bullet^1}(Y_1),$$

where $\Gamma(D_1 \cdot Y_1, \dots, D_r \cdot Y_1) \subseteq \overline{\text{Eff}}(Y_1)$ is the convex cone generated by the numerical equivalence classes of the divisors $D_1 \cdot Y_1, \dots, D_r \cdot Y_1$ on Y_1 . \square

Remark 3.7. It should be noted that the above result does not hold for general admissible flags of subvarieties of a Mori dream space. Indeed, [KLM12, Example 3.4] shows that $X := \mathbb{P}^2 \times \mathbb{P}^2$ can be equipped with an admissible flag Y_\bullet such that the Okounkov body $\Delta_{Y_\bullet}(D)$, where D is a divisor in the linear series $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(3, 1)$, is not polyhedral. The flag Y_\bullet here is of course not a flag of Mori dream spaces: if $Y_1 := \mathbb{P}^2 \times E$, where $E \subseteq \mathbb{P}^2$ is a general elliptic curve, were a Mori dream space, then its image E under the second projection would also be a Mori dream space by [Oka11, Theorem 1.1]. However, \mathbb{P}^1 is the only Mori dream curve (cf. [C12, p. 6]).

Finally, [KLM12, Prop. 3.5] gives another example of a non-polyhedral Okounkov body of a divisor on a Mori dream space Z with respect to a family of admissible flags Y_\bullet . However, the description of the pseudo-effective cone $\overline{\text{Eff}}(Y_1)$ shows that this cone is defined by quadratic equations and is thus not polyhedral; hence the divisor Y_1 on Z is not a Mori dream space.

4. OKOUNKOV BODIES ON BOTT-SAMELSON VARIETIES

In this section we apply the general results from the previous section to Bott-Samelson varieties.

4.1. Global Okounkov bodies. On a Bott-Samelson variety Z_w defined by a reduced sequence w let Y_\bullet be the natural vertical flag described in section 2.1. In order to apply Theorem 3.6 we will show that Y_\bullet is in fact a good flag. The fact that each Y_i is a Mori dream space follows from Theorem 2.1 and the second condition is a consequence of the following proposition together with the fact that the exceptional locus of a SQM equals the stable base locus of the defining movable divisor.

Proposition 4.1. *Let D be a Cartier divisor on Z_ω , and F a base component of the complete linear series $|D|$. Then F intersects $Y := Y_1$ properly, i.e.,*

$$\text{codim}_Y(F|_Y) = \text{codim}_X(F)$$

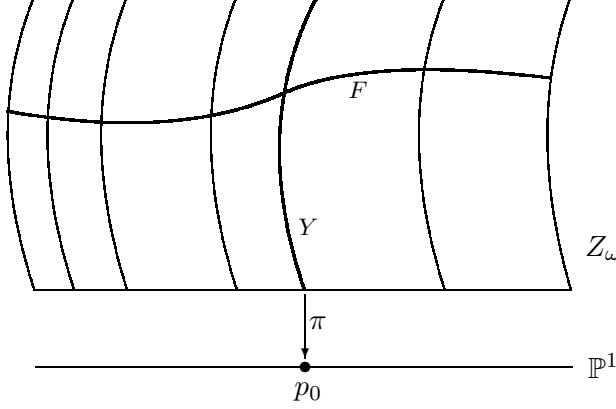
Proof. We first prove that the base locus of $|D|$ is invariant under the group P_1 . Let therefore $s \in H^0(X, \mathcal{O}_X(D))$ be a section, and let $p \in P_1$. Then for any $x \in Z_\omega$

$$s(px) = pp^{-1}s(px) = p(p^{-1}s(px)).$$

By (4), the right hand side is exactly $p((p^{-1}s)(x))$ and $(p^{-1}s \in H^0 X, \mathcal{O}_X(D))$ vanishes in x if x is in the base locus $B(D)$. Therefore,

$$s(px) = p0_{L_x} = 0_{L_{px}},$$

and the claim follows. Then, every irreducible component of $B(D)$ is also P_1 -invariant. In particular, F is invariant under P_1 .



Now, the P_1 -action on \mathbb{P}^1 has only one orbit, so P_1 operates transitively on the fibres of π . Hence, the base component F is given by the union of orbits of elements in the restriction $F|_Y$. Therefore, the generic fibre dimension holds for all fibres of π , so that

$$\dim F = \dim F|_Y + 1,$$

which implies the statement. \square

Theorem 4.2. *Let Z_w be a Bott-Samelson variety defined by a reduced sequence w . Then the global Okounkov body Δ_{Y_\bullet} is rational polyhedral.*

Proof. By Theorem 2.1, the variety Z_w is a Mori dream space. The same is true for each Y_i . Moreover, by construction, Y_i defines a Cartier divisor on Y_{i-1} and is cut out by a global section s_i which is just the pullback of a section $t_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ vanishing in the B -fixed point p_0 . Inductive application of Proposition 4.1 then shows that Y_\bullet is a good flag and the result follows from Theorem 3.6. \square

We can now also show that the Okounkov bodies of effective line bundles over Schubert varieties, with respect to a natural valuation-like function, are rational polyhedral. Indeed, Schubert varieties have rational singularities, so that the projection morphism $p_w : Z_w \rightarrow Y_{\overline{w}}$ satisfies the property $(p_w)_* \mathcal{O}_{Z_w} = \mathcal{O}_{Y_{\overline{w}}}$ (cf. [B, Section 2.2]). Hence, for any effective line bundle L on $Y_{\overline{w}}$, we have

$$(10) \quad H^0(Y_{\overline{w}}, L) \cong H^0(Z_w, p_w^* L).$$

Let now

$$\nu : \text{Cox}(Z_w)_h \setminus \{0\} \rightarrow \mathbb{N}_0^n,$$

where $\text{Cox}(Z_w)_h$ denotes the set of homogeneous elements in the Cox ring of Z_w with respect to the effective basis, be the valuation-like function defined by the flag Y_\bullet , and let

$$\nu_L : \bigsqcup_{k \geq 0} H^0(Y_{\overline{w}}, L^k) \setminus \{0\} \rightarrow \mathbb{N}_0^n$$

be the valuation-like function naturally defined by the isomorphisms (10) (for all powers L^k) and restriction of ν . Then, the Okounkov body $\Delta_{\nu_L}(L)$

coincides with the slice $p_2^{-1}(p_w^*L) \cap \Delta_{Y_\bullet}(X)$ of $\Delta_{Y_\bullet}(X)$, and hence is rational polyhedral. Thus, we have proved the following corollary.

Corollary 4.3. *Let L be an effective line bundle over the Schubert variety $Y_{\overline{w}}$ of G/B . Then, the Okounkov body $\Delta_{Y_\bullet}(L)$ defined by the natural valuation-like function v_L defined by the flag Y_\bullet in Z_w is a rational polytope.*

If $Y_{\overline{w}} = G/B$ is a flag variety, the Picard group $\text{Pic}(G/B)$ has an effective basis, namely the line bundles $L_i = G \times_{\omega_i} \mathbb{C}$ defined by the fundamental weights ω_i , $i = 1, \dots, r$, with respect to a choice of simple roots for the root system of G . Let $\Sigma \subseteq \overline{\text{Eff}}(Z_w)$ be the closed convex cone generated by the divisors of the line bundles $p_w^*L_i$, $i = 1, \dots, r$. By the isomorphisms (10) we now have

$$\Delta_{Y_\bullet}(G/B) \cong p_2^{-1}(\Sigma) \cap \Delta_{Y_\bullet}(Z_w).$$

Since the cone Σ is finitely generated, the cone on the right hand side is rational polyhedral, so that we have proved the following corollary.

Corollary 4.4. *The global Okounkov body $\Delta_{Y_\bullet}(G/B)$ of the flag variety G/B , with respect to the valuation defined by the flag Y_\bullet of subvarieties of Z_w , is a rational polyhedral cone.*

4.2. Weight multiplicities. We now turn our attention to the action of a torus $H \subseteq B$, contained in a maximal torus of G lying in B , on the section ring $R(D) := \bigoplus_{k \geq 0} H^0(Z_w, \mathcal{O}_{Z_w}(kD))$ of an effective divisor D on Z_w . Recall that each section space $H^0(Z_w, \mathcal{O}_{Z_w}(kD))$ carries a representation of B given by the action of B as automorphisms of the line bundle $\mathcal{O}_{Z_w}(kD)$ (cf. [LT04]). Moreover, the flag Y_\bullet consists of B -invariant subvarieties of Z_w , so that the valuation-like function

$$\nu_D : \bigsqcup_{k \geq 0} H^0(Z_w, \mathcal{O}_{Z_w}(kD)) \setminus \{0\} \longrightarrow \mathbb{N}_0^n$$

is B -invariant, i.e., the identity $\nu(b.s) = \nu(s)$ holds for any non-zero section $s \in H^0(Z_w, \mathcal{O}_{Z_w}(kD))$, and $b \in B$. Hence, there is a well-defined projection

$$q : \Delta_{Y_\bullet}(D) \longrightarrow \Pi_D$$

onto the weight polytope (cf. [B86]) of the section ring $R(D)$ for the action of the torus H (cf. [Oko96] [KK10]). If $\mathfrak{h} = \text{Lie}(H)$ is the Lie algebra of H , and the $\mu \in \Pi_D \subseteq \mathfrak{h}$ is a rational point in the interior of the weight polytope, we then have that the asymptotics of the weight spaces $W_{k\mu} \subseteq H^0(Z_w, \mathcal{O}_{Z_w}(kD))$ are given by

$$\lim_{k \rightarrow \infty} \frac{\dim W_{k\mu}}{k^{d-r}} = \text{vol}_{d-r}(q^{-1}(\mu) \cap \Delta_{Y_\bullet}(D)),$$

where r is the dimension of the moment polytope Π_D , d is the dimension of the Okounkov body $\Delta_{Y_\bullet}(D)$ (and which equals the Iitaka dimension of the line bundle $\mathcal{O}_{Z_w}(D)$), and the right hand side denotes the $(d-r)$ -dimensional Lebesgue measure of the slice $q^{-1}(\mu) \cap \Delta_{Y_\bullet}(D)$ of the Okounkov body $\Delta_{Y_\bullet}(D)$. We thus get the following result, saying the the asymptotics of weight spaces are given by polyhedral expressions.

Corollary 4.5. *For any effective divisor D on Z_w , and rational point $\mu \in \Pi_D$ in the interior of Π_D , the asymptotic multiplicity*

$$\lim_{k \rightarrow \infty} \frac{\dim W_{k\mu}}{k^{d-r}}$$

is the volume of a rational polytope. As a consequence, the same holds for the weight spaces

$$W_{k\mu} \subseteq H^0(Y_{\overline{w}}, L^k)$$

for an effective line bundle L over a Schubert variety $Y_{\overline{w}}$.

Proof. We only need to prove the second claim about Schubert varieties. Here we notice that the projection morphism $p_w : Z_w \rightarrow Y_{\overline{w}}$ is B -equivariant, and so in particular H -invariant. Hence, the isomorphisms (10) for the powers L^k are H -equivariant, so that the claim thus follows from the first part about Bott-Samelson varieties. \square

REFERENCES

- [AS14] Anderson, D., Stapledon, A., *Schubert varieties are log Fano over the integers*, Proc. Amer. Math. Soc. **142** (2014), 2, 409–411
- [B86] Brion, M., *Sur l'image de l'application moment*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986), Lecture Notes in Math., **1296**, 177–192, Springer 1987
- [B] Brion, M., *Lectures on the geometry of flag varieties*, lecture notes
- [C12] Casagrande, C., *Mori dream spaces and Fano varieties*, lecture notes, 2012
- [HK00] Hu, Y., Keel, S., *Mori Dream Spaces and GIT*, Michigan Math. J. **48** (2000), 331–348
- [KK09] Kaveh, K., Khovanskii, A.G., *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, Ann. of Math. (2) **176** (2012), 2, 925–978
- [KK10] Kaveh, K., Khovanskii, A.G., *Convex bodies associated to actions of reductive groups*, Mosc. Math. J. **12** (2012), 2, 369–396, 461
- [K11] Kaveh, K., *Crystal bases and Newton-Okounkov bodies*, preprint, 2011, <http://arxiv.org/abs/1101.1687>
- [KLM12] Kőröny, A., Lozovanu, V., Maclean, C., *Convex bodies appearing as Okounkov bodies of divisors*, Adv. Math. **229** (2012), 2622–2639
- [LT04] Lauritzen, N., Thomsen, J.F., *Line bundles on Bott-Samelson varieties*, J. Alg. Geom. **13** (2004), 461–473
- [LM09] Lazarsfeld, R., Mustață, M., *Convex bodies associated to linear series*, Ann. Sci. Éc. Norm. Supér. **42** (2009), 783–835
- [Li98] Littelmann, P., *Cones, crystals, and patterns*, Transform. Groups **3** (1998), 2, 145–179
- [Oka11] Okawa, S., *On images of Mori dream spaces*, preprint, 2011, arxiv.org/abs/1104.1326
- [Oko96] Okounkov, A., *Brunn-Minkowski inequality for multiplicities*, Invent. Math. **125** (1996), 405–411
- [SS14] Schmitz, D., Seppänen, H., *On the polyhedrality of global Okounkov bodies*, arxiv.org/abs/1403.4517, to appear in *Adv. Geom.*

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